

$U(3)$ artificial gauge fields for cold atoms

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We propose to generate an artificial non-Abelian $U(3)$ gauge field by using a 2-tripod scheme, namely two tripod configurations sharing a common ground state level and driven by resonant 1-photon transitions. Using an appropriate combination of four Laguerre-Gauss and two Hermite-Gauss laser beams, we are able to produce a $U(3)$ -monopole and a $U(3)$ spin-orbit coupling for both alkali and alkaline-earth atoms. This 2-tripod scheme could open the way to the study of interacting spinor condensates subjected to $U(3)$ -monopoles.

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I. INTRODUCTION

Within less than a decade, ultracold quantum gases have successfully pervaded many fields of physics. Indeed, they provide a rather unique testing bed where theorists' dreams can be turned into carefully designed experimental situations. This is particularly true in the condensed matter realm where they became a key player in many-body physics [1–3]. Quantum Hall effects did not escape the trend. The catch however is that atoms are neutral and one thus needs to implement an artificial gauge field acting on the atoms that would give rise to a strong enough effective magnetic field. A first idea was to set quantum gases into rapid rotation [4]. Since then, more versatile and promising schemes have been introduced, some even realized, all based on light-atom interactions [5–11]. These light-induced artificial gauge fields, encompassing Abelian and non-Abelian situations, have opened the door to a whole class of model Hamiltonians [12–15] and are addressing diverse physical situations ranging from artificial Dirac monopoles [16], spin-orbit (SO) coupling [17, 18] and topological phases [19, 20], to non-Abelian particles [21], and mixed dimensional systems [22–26].

In this Letter, we discuss a new proposal to generate artificial non-abelian $U(3)$ gauge fields. Our scheme is a straightforward generalization of the tripod scheme discussed in [27]. It is based on three space-dependent dark states arising from the coupling with resonant one-photon transitions between Zeeman sub-levels belonging to different hyperfine states of an alkali atom, such as ^{87}Rb , subjected to a magnetic field. In the following, we first introduce the laser scheme we propose and work out the general expressions for both the effective vector and scalar fields. We next discuss two specific laser configurations: the first one gives rise to a non-Abelian $U(3)$ monopole while the second one gives rise to a non-Abelian SO-like coupling. Finally, we discuss alkaline-earth atoms, taking the fermionic isotope of Strontium

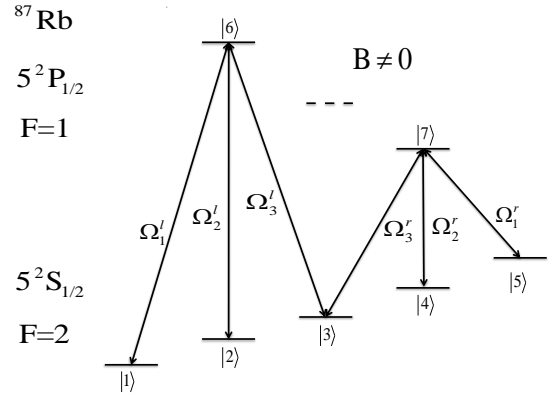


Figure 1. 2-tripod scheme for the D_1 line of ^{87}Rb . An external magnetic field, chosen as the quantization axis, is applied to the atoms and lifts the Zeeman degeneracy of the ground state (with total spin $F_g = 2$) and excited state (with total spin $F_e = 1$) manifolds. Six resonant laser beams then illuminate the atoms, their polarization state being appropriately chosen to address the π and σ_{\pm} transitions shown in the Figure. States $|i\rangle$ ($1 \leq i \leq 5$) refer to the Zeeman levels $|2, m_g\rangle$ ($|m_g| \leq 2$) in the ground state manifold. States $|i\rangle$ ($i = 6, 7$) refer to the Zeeman levels $|1, m_e = \pm 1\rangle$ of the excited states manifold. All other excited levels (three of them indicated by horizontal dashed lines) are not addressed since they are far off-resonance from any of the six laser beams. This 2-tripod scheme gives rise to three dark states with vanishing energy.

as a paradigmatic example. In this case however, because the Zeeman shifts of the lowest hyperfine states 1S_0 are negligible, a slightly different laser configuration is required to appropriately couple the electronic levels.

II. 2-TRIPOD SCHEME

We consider the 2-tripod coupling scheme depicted in Fig. 1. It is based on two usual tripod schemes. One couples ground states $|1\rangle, |2\rangle, |3\rangle$ to excited level $|6\rangle$. The other one couples ground states $|3\rangle, |4\rangle, |5\rangle$ to the excited level $|7\rangle$. As one can see these two tripod schemes are not independent since they share one common ground state level, namely level $|3\rangle$. This very situation can be implemented with alkali atoms, for instance by considering the D_1 line of ^{87}Rb atoms. In this case, one first applies a magnetic field to split the Zeeman structure of both the ground $F_g = 2$ and excited $F_e = 1$ states and then one shines six suitably polarized resonant laser beams to produce the desired 2-tripod coupling scheme shown in Fig. 1. Sec. V below gives more details about the experimental realization and its limitations.

Since the 2-tripod scheme involves five ground states coupled to two excited states, one expects three degenerate dark states (DS) with vanishing energy. Indeed, in the rotating wave approximation, the 2-tripod Hamiltonian reads:

$$H_0 = -\hbar(\Omega_1^l|6\rangle\langle 1| + \Omega_2^l|6\rangle\langle 2| + \Omega_3^l|6\rangle\langle 3| + \Omega_3^r|7\rangle\langle 3| + \Omega_2^r|7\rangle\langle 4| + \Omega_1^r|7\rangle\langle 5|) + h.c. \quad (1)$$

We now parametrize the position-dependent Rabi frequencies as follows [12, 27]:

$$\begin{aligned} \Omega_1^a &= \Omega_a \sin \theta_a \cos \phi_a e^{iS_1^a} \\ \Omega_2^a &= \Omega_a \sin \theta_a \sin \phi_a e^{iS_2^a} \\ \Omega_3^a &= \Omega_a \cos \theta_a e^{iS_3^a}, \end{aligned} \quad (2)$$

where $a = l, r$. The twelve different quantities Ω_i^a , θ_a , ϕ_a and S_i^a ($i = 1, 2, 3$) are generally space-dependent. It is then straightforward to compute the three orthonormal DS of the 2-tripod scheme:

$$|D_l\rangle = \sin \phi_l e^{iS_{31}^l} |1\rangle - \cos \phi_l e^{iS_{32}^l} |2\rangle \quad (3a)$$

$$|D_r\rangle = \sin \phi_r e^{iS_{31}^r} |5\rangle - \cos \phi_r e^{iS_{32}^r} |4\rangle \quad (3b)$$

$$|D_0\rangle = \frac{1}{\alpha_0} \left[\cot \theta_l \cos \phi_l e^{iS_{31}^l} |1\rangle + \cot \theta_l \sin \phi_l e^{iS_{32}^l} |2\rangle - |3\rangle + \cot \theta_r \cos \phi_r e^{iS_{31}^r} |5\rangle + \cot \theta_r \sin \phi_r e^{iS_{32}^r} |4\rangle \right]$$

$$\alpha_0 = (1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2} \quad (3c)$$

where $S_{ij}^a = S_i^a - S_j^a$ ($a = l, r$). One may note that when $\Omega_3^r = 0$, corresponding to $\theta_r = \pi/2$, then the 2-tripod scheme breaks up into a $U(2)$ tripod configuration coupling states $|1\rangle, |2\rangle, |3\rangle$ to $|6\rangle$ and an independent $U(1)$ Λ -configuration coupling states $|4\rangle, |5\rangle$ to $|7\rangle$. Then state $|D_r\rangle$ identifies with the DS of the Λ -configuration while $|D_l\rangle$ and the corresponding $|D_0\rangle$ state identify with the two DS of the left-tripod configuration. The same type of considerations can be made if $\Omega_3^l = 0$, corresponding to $\theta_l = \pi/2$. In other words, the states $|D_a\rangle$ ($a = l, r$)

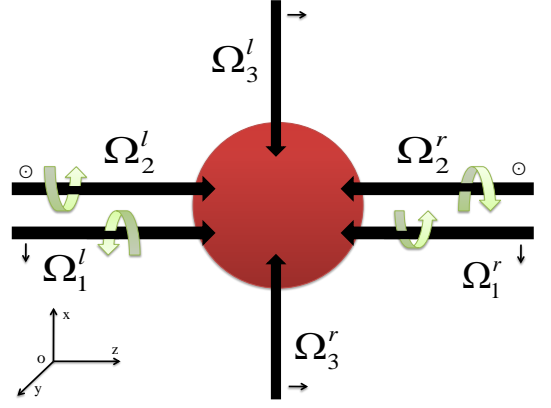


Figure 2. [Color online] Laser beam configuration giving rise to a non-Abelian $U(3)$ -monopole with unit charge and associated to the generator J_x of the $SO(3)$ subgroup. The horizontal beams are Laguerre-Gauss modes carrying opposite orbital angular momentum $\pm \hbar$ shown by the green arrows. The vertical laser beams are $n = 1$ Hermite-Gauss modes. The bias magnetic field lifting the Zeeman degeneracies in Fig. 1 defines the quantization axis. One possible configuration consists in choosing the magnetic field along Oy . Then, all beams are linearly-polarized (thin black arrows) and can selectively address their allocated transitions since their polarization state have a non-vanishing projection on the desired σ_{\pm} and π transitions.

are DS for the left and right $U(2)$ tripod configuration as well as DS for the Λ configuration. The remaining state $|D_0\rangle$, embodying all five Zeeman ground state levels, reflects the coupling of the two $U(2)$ tripod configurations when both Ω_3^l and Ω_3^r are non zero. It boils down to the missing tripod DS when Ω_3^l or Ω_3^r vanishes.

From the DS expressions (3), one can derive the vector and scalar potentials associated to the 2-tripod scheme. The vector potential \vec{A} is now a 3×3 Hermitian matrix with entries:

$$\begin{aligned} \vec{A}_{11} &= \cos^2 \phi_l \vec{\nabla} S_{23}^l + \sin^2 \phi_l \vec{\nabla} S_{13}^l \\ \vec{A}_{33} &= \cos^2 \phi_r \vec{\nabla} S_{23}^r + \sin^2 \phi_r \vec{\nabla} S_{13}^r \\ \vec{A}_{22} &= \frac{1}{\alpha_0^2} \left[\cot^2 \theta_l (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l) + \cot^2 \theta_r (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r) \right] \\ \vec{A}_{21} &= \vec{A}_{12}^* = \frac{\cot \theta_l}{\alpha_0} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l + i \vec{\nabla} \phi_l \right) \\ \vec{A}_{23} &= \vec{A}_{32}^* = \frac{\cot \theta_r}{\alpha_0} \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r + i \vec{\nabla} \phi_r \right) \\ \vec{A}_{31} &= \vec{A}_{13}^* = 0, \end{aligned} \quad (4)$$

where the star denotes complex conjugation. The scalar potential expression is rather involved and is given in the Appendix A for sake of completeness.

III. $U(3)$ MONOPOLE

In the following, we will use the spherical coordinate system (r, θ, φ) about axis Oz to parametrize a point $M(x, y, z)$ in space. Then, from Eq. (4), one can check that a gauge field corresponding to a $U(3)$ monopole can be generated by using the following Rabi frequencies:

$$\begin{aligned}\Omega_{1,2}^l &= \Omega_0 \frac{\rho}{R} e^{i(kz \mp \varphi)} & \Omega_3^l &= \Omega_0 \frac{z}{R} e^{ikx} \\ \Omega_{1,2}^r &= \Omega_0 \frac{\rho}{R} e^{-i(kz \pm \varphi)} & \Omega_3^r &= \Omega_0 \frac{z}{R} e^{-ikx},\end{aligned}\quad (5)$$

where $\rho = \sqrt{x^2 + y^2} = r \sin \theta$. The corresponding laser beam configuration is shown in Fig. 2, the quantization axis being along axis Oy . The three beams addressing the left tripod configuration consist of two linearly-polarized co-propagating (along axis Oz) Laguerre-Gauss beams with orbital angular momentum $\pm \hbar$ and of a linearly-polarized Hermite-Gauss beam propagating along axis Ox . The three remaining beams are just "reflection images" of the previous beams and address the right tripod configuration. The potential vector then reads:

$$\begin{aligned}\vec{A} &= -\hbar \frac{\cos \theta}{\sqrt{2} \sin \theta} \frac{\hat{e}_\varphi}{r} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &\quad - \hbar k \sin^2 \theta (\hat{e}_z - \hat{e}_x) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + \hbar k (\hat{e}_z - \hat{e}_x) \mathbb{1}\end{aligned}\quad (6)$$

The last term is inessential: it is a constant gradient term proportional to the unit matrix that can be gauged away through a $U(1)$ transformation. The second term depends on the wave number k and is similar to the k -dependent terms found in the $U(2)$ monopole case [12, 27]. It is not singular and leads to non-monopole terms. We will not discuss it in the following though it can play an important role in the dynamics [19]. Finally, the first term can be rewritten as $\vec{A}_m = \vec{a}_m J_x$ where $\vec{a}_m = -\cos \theta \hat{e}_\varphi / (r \sin \theta)$. It corresponds to a non-Abelian $U(3)$ monopole with unit effective magnetic charge $Q = 1$ coupled to the spin-1 operator

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (7)$$

Indeed, the corresponding non-Abelian magnetic field $\vec{B}_m = \vec{\nabla} \times \vec{A}_m + \vec{A}_m \times \vec{A}_m / (i\hbar)$ reads $\vec{B}_m = (\vec{\nabla} \times \vec{a}_m) J_x$. Let us first consider a general Abelian vector field of the form $\vec{a} = g(\theta) \hat{e}_\varphi / (r \sin \theta)$. Then the corresponding Abelian magnetic field reads:

$$\vec{B} = \vec{\nabla} \times \vec{a} = \frac{g'(\theta)}{\sin \theta} \frac{\hat{e}_r}{r^2} - 2\pi \phi(z) \delta(x) \delta(y) \hat{e}_z \quad (8)$$

$$\phi(z) = g(0)\Theta(z) + g(\pi)\Theta(-z), \quad (9)$$

where $\Theta(u)$ is the step function. It consists of a monopole contribution B_m given by the first term in the right-hand side of Eq.(8) and of a Dirac-like string contribution with flux $-2\pi\phi(z)$. The magnetic charge Q_m associated to the monopole field is computed with the help of Gauss theorem. It reads:

$$Q_m = \frac{1}{4\pi} \int_S \vec{B} \cdot d\vec{S} = \frac{g(\pi) - g(0)}{2}, \quad (10)$$

where S is the sphere of radius r and $d\vec{S} = r^2 \sin \theta d\theta d\varphi \hat{e}_r$ [29]. Since we have $g(\theta) = -\cos \theta$ in our 2-tripod situation, we thus get a genuine monopole field with unit magnetic charge coupled to J_x . It is with noticing that, in high-energy physics, it has been established that *all* non-Abelian monopoles are simply obtained as Abelian monopoles times a *constant* (charge) matrix Q [30]. The string is undetectable when $\exp(4\pi i Q / \hbar) = \mathbb{1}$. This is also what we get here.

If, starting from the laser configuration shown in Fig. 2, one just flips the sign of the orbital angular momentum carried by each of the right Laguerre-Gauss fields, i. e.

$$\Omega_{1,2}^r = \Omega_0 \frac{\rho}{R} e^{-i(kz \mp \varphi)}, \quad (11)$$

while keeping all the other fields unaffected (see Fig 3), then only the monopole part of the full $U(3)$ vector potential is modified and now reads $\vec{A}_m = \vec{a}_m \tilde{J}_x$ where:

$$\tilde{J}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = S J_x S, \quad (12)$$

where S is the diagonal matrix with entries $(1, 1, -1)$ and representing a reflection about the plane (Ox, Oy) . The Abelian monopole with unit charge described by \vec{a}_m is now coupled to the new matrix \tilde{J}_x . This monopole is thus associated with a Hermitian matrix which turns out to be a generator, like J_x is, of the rotation subgroup $SO(3)$ of $U(3)$ (see later discussion). One may note that $J_x = \sqrt{2}\hbar(g_1 + g_6)$ while $\tilde{J}_x = \sqrt{2}\hbar(g_1 - g_6)$, where the matrices $g_{1,6}$ are Gell-Mann matrices [28]. For sake of completeness, we give the 8 Gell-Mann matrices g_i ($1 \leq i \leq 8$) in the Appendix. By changing the laser beams configuration, a $U(3)$ monopole associated to another combination of Gell-Mann matrices could be obtained in principle.

Similarly to the $U(2)$ situation studied in [19], understanding the topological properties [31] of the ground state and excitations of interacting particles subjected to a $U(3)$ monopole field would certainly lead to new and interesting physics that could be targeted in cold atoms experiments. More precisely, since the $U(3)$ gauge group is larger than $U(2)$, one expects that different kind of topological charges show up, depending on the symmetry of the ground state.

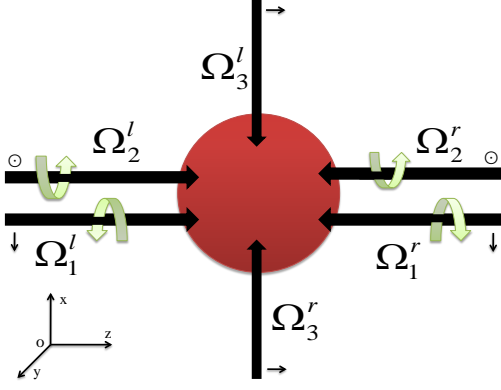


Figure 3. [Color online] Laser beams configuration generating a non-Abelian $U(3)$ -monopole with unit charge and associated to a 3×3 matrix which does not belong to the $SO(3)$ subgroup. It is obtained by flipping the sign of the orbital angular momentum carried by the right horizontal beams. The conventions and polarizations of the beams are the same as in Fig. 2.

IV. SPIN-ORBIT COUPLING

A non-Abelian SO-coupling can alternatively be achieved by considering the laser beams configuration shown in Fig. 2 but where all six laser beams are now linearly-polarized plane waves with the following Rabi frequencies:

$$\begin{aligned} \Omega_{1,2}^l &= \frac{\Omega_l}{\sqrt{2}} \sin \theta_l e^{\pm i \vec{k}_l \cdot \vec{r}}, & \Omega_3^l &= \Omega_l \cos \theta_l e^{-i \vec{k}_3 \cdot \vec{r}}, \\ \Omega_{1,2}^r &= \frac{\Omega_r}{\sqrt{2}} \sin \theta_r e^{\pm i \vec{k}_r \cdot \vec{r}}, & \Omega_3^r &= \Omega_r \cos \theta_r e^{i \vec{k}_3 \cdot \vec{r}}, \end{aligned} \quad (13)$$

($\varphi_r = \varphi_l = \pi/4$). Here we have $\vec{k}_3 \perp \vec{k}_{l,r}$, these vectors being in the plane (Ox, Oz) orthogonal to the quantization axis Oy , the polarization states being the same as in Fig. 2. Simplifying further to the case $\theta_r = \theta_l = \Theta$, the effective vector potential reads:

$$\vec{A} = \frac{1}{2} \vec{k}_3 J_z + \frac{2\hbar}{\sqrt{2 + \tan^2 \Theta}} (\vec{k}_l g_1 + \vec{k}_r g_6) \quad (14)$$

$$= \frac{1}{2} \vec{k}_3 J_z + \frac{2\hbar \vec{k}_l}{\sqrt{2 + \tan^2 \Theta}} (g_1 \pm g_6), \quad (15)$$

where the last equality has been obtained by also assuming $\vec{k}_r = \pm \vec{k}_l$. One can generate different types of SO-coupling, in the spirit of what is done in [32] to induce a Rashba or a Linear Dresselhaus SO-coupling. For example, from the expansion of $(\vec{p} - \vec{A})^2/(2m)$, we get the SO coupling terms $V_{SO} = \hbar k/(2m) v_{SO}$ where

$$\begin{aligned} v_{SO} &= (J_z/\hbar) p_x + 2\beta(\Theta)(g_1 \pm g_6) p_z \\ v_{SO} &= 2\beta(\Theta)(g_1 \pm g_6) p_x + (J_z/\hbar) p_z \end{aligned} \quad (16)$$

for $(\vec{k}_3, \vec{k}_l) = (k\hat{e}_x, k\hat{e}_z)$ and for $(\vec{k}_3, \vec{k}_l) = (k\hat{e}_z, k\hat{e}_x)$. Here $\beta(\Theta) = 2/\sqrt{2 + \tan^2 \Theta}$ and equals unity if $\tan \Theta = \sqrt{2}$, i. e. if all left (resp. right) Rabi frequencies are equal. One should note that, contrary to the $U(2)$ scheme, the expansion features the non-Abelian potential $\vec{A} \cdot \vec{A}/(2m)$ which adds up to the non-Abelian potential Φ . As one can easily check, this term is not proportional to the identity and therefore should play a role in the dynamics and in the ground state properties of an interacting system. For $\tan \Theta = \sqrt{2}$, it reads:

$$\frac{\vec{A} \cdot \vec{A}}{2m} = \frac{\hbar^2 k^2}{4m} (\mathbb{1} - g_4). \quad (17)$$

V. EXPERIMENTAL REALIZATION AND LIMITATIONS

In Sec. III, we have shown that non-Abelian $U(3)$ -monopole contributions can be obtained using specific laser beams configurations. More precisely, the common ground state shared by the two tripods, namely ground state level $|3\rangle$, should be coupled to excited states with Hermite-Gauss laser beams propagating along the same axis. The other ground states are coupled to excited states with Laguerre-Gauss beams propagating perpendicularly to the Hermite-Gauss beams, see Eqs. (5),(11) and Figs. 2,3. Because of these constraints, one cannot solely rely on the laser polarization degrees of freedom and dipole selection rules to independently and selectively address the different transitions of the 2-tripod scheme. One also needs to apply a strong magnetic field to lift the Zeeman degeneracy in the ground state and excited state manifolds to get well separated transitions and avoid spurious spontaneous emission processes from unwanted transitions.

For the D_1 line of ^{87}Rb that we used as an example, this Zeeman degeneracy lifting is even favored by the opposite signs of the Landé factors in the ground state ($F_g = 2$) and in the excited state ($F_e = 1$). Regardless any possible technical issues to generate the required magnetic field, its maximum value is limited by the hyperfine splitting $\Delta_{hs} \approx 142\Gamma$ of the excited state ($\Gamma = 2\pi \times 5.8$ MHz is the natural line width of the transition), otherwise the coupling to the other hyperfine manifold $F_e = 2$ will start playing a non-negligible role. In addition, since light-induced gauge potentials originate from photon momentum exchanges with the atoms, their energy scale is thus of the order of the recoil energy, $E_R = \hbar\omega_R \approx 0.6 \cdot 10^{-3} \hbar\Gamma$. Therefore, the rate of any residual spontaneous emission from the off-resonant transitions should be smaller than $\hbar^{-1} E_R$, which, in the case of ^{87}Rb , might be difficult to achieve. In the same line of thought, it becomes even more challenging to address the effect of the gauge potential on the atoms dynamics over time scales in the millisecond range and beyond [6].

From this point of view, fermionic isotopes of alkaline-earth atoms provide interesting alternatives to Rubidium

atoms. In particular, the hyperfine splitting of the 3P_1 excited state of the Strontium isotope ^{87}Sr is $\Delta_{hs} \approx 10^5\Gamma$, at least three orders of magnitude larger than alkali atoms. In this case, the left and right tripod configurations can be driven independently (see Fig. 4) and should be well protected from spurious spontaneous emission due to unwanted off-resonant transitions. Furthermore the narrow linewidth ($\Gamma = 7.4$ kHz) of the intercombination line at 689nm leads to a large Zeeman shift (compared to Γ) with reasonable magnetic fields of few tens of Gauss. However the 1S_0 ground state carries no electronic spin, only a nuclear one $I = 9/2$. Regarding our laser coupling scheme, the Zeeman shift in the ground state is thus essentially negligible and one has now to solely rely on the polarization states of the beams to address independently and selectively the left and right tripod transitions. More precisely, the Laguerre-Gauss beams addressing the left (resp. right) tripod, and propagating along the Oz axis, must have opposite circular polarizations and should now drive the $1 \leftrightarrow 6$ and $3 \leftrightarrow 6$ σ_{\pm} -transitions (resp. the $3 \leftrightarrow 7$ and $5 \leftrightarrow 7$ σ_{\pm} -transitions). The Hermite-Gauss beams, propagating along the Ox axis, must have a linear polarization, along the Oz axis, and address the $2 \leftrightarrow 6$ and $4 \leftrightarrow 7$ π -transitions. One immediately sees that the laser beams configuration proposed in Sec. III, and generating an $U(3)$ -monopole, fails to meet these polarization constraints. Figure 4 shows the new laser beams configuration with the appropriate quantization axis and polarization states. In the following Sec. VI, we discuss in some details the properties of the resulting new gauge potentials. In particular, we show that a $U(3)$ -monopole with a non-zero charge can still be recovered by using an appropriate gauge transformation.

Finally, let us mention that the SO-coupling scheme, depicted in Sec. IV, can be extended to the case of the alkaline-earth atoms, again using appropriate polarization states for the laser beams to address the proper transitions. One finds SO-coupling terms similar to those given in Eq (16).

VI. ALKALINE-EARTH ATOMS

A. $U(3)$ monopole

Taking into account the polarization constraints that alkaline-earth atoms bring into the game, a laser configuration that corresponds to a realistic experimental situation is the following:

$$\Omega_{1,3}^l = \Omega_0 \frac{\rho}{R} e^{i(kz \mp \varphi)}, \quad \Omega_2^l = \Omega_0 \frac{z}{R} e^{ikx}. \quad (18)$$

$$\Omega_{1,3}^r = \Omega_0 \frac{\rho}{R} e^{-i(kz \mp \varphi)}, \quad \Omega_2^r = \Omega_0 \frac{z}{R} e^{-ikx}. \quad (19)$$

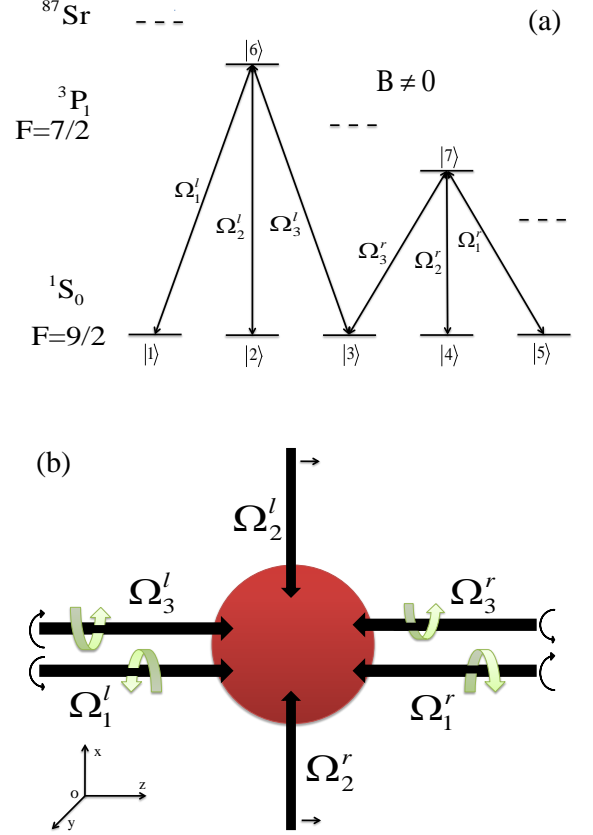


Figure 4. [Color online] (a) 2-tripod scheme in the case of ^{87}Sr atoms. An external magnetic field is still applied but, for reasonable magnetic strength, only the Zeeman degeneracy of the excited state manifold is lifted (the ground state manifold carries no electronic spin). Now, contrary to the Rubidium case, the laser fields must have exactly the polarization required by the transition they need to address. This imposes the Zeeman field to be applied along the quantization axis Oz , the left and right Laguerre-Gauss beams to propagate along Oz with opposite circular polarizations (shown with the thin black arrows) to address the σ_{\pm} -transitions. The vertical Hermite-Gauss beams propagate along Ox and are linearly-polarized along Oz to address the π -transitions, see (b).

The corresponding non-Abelian vector potential is:

$$\vec{A} = \hbar \frac{\cos \theta \sin \theta}{\sqrt{1 + 2 \sin^2 \theta}} \frac{\hat{e}_\varphi}{r} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \hbar \frac{(2 - \sin^2 \theta)}{\sin \theta} \frac{\hat{e}_\varphi}{r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + (\dots), \quad (20)$$

where (\dots) represents terms which do not contribute to the singular part of the radial magnetic field such as (non-singular) k -dependent terms or the (φ -independent) \hat{e}_θ component. Each term explicitly written in Eq. (20) cor-

respond to a non-Abelian monopole-like magnetic field, the first one coupled to $2\hbar(g_1 - g_6)$ and the second one to J_z (which is also a linear combination of the Gell-Mann matrices of the $SU(3)$ group). However, each of these terms alone are such that $g(\pi) = g(0)$. They thus each carry a vanishing magnetic charge according to Eq. (10) and the present configuration seems not, strictly speaking, to produce a true magnetic monopole. However, let us apply the gauge transformation $U = e^{i\theta \tilde{J}_y/\hbar}$ where

$$\tilde{J}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} = S J_y S. \quad (21)$$

Since $\tilde{J}_a = S J_a S$ ($a = x, y, z$) satisfy the usual angular momentum commutation relations $[\tilde{J}_a, \tilde{J}_b] = i\hbar \varepsilon_{abc} \tilde{J}_c$ (note that $\tilde{J}_z = S J_z S = J_z$), one can use the rotation algebra and easily compute the transformed monopole gauge field $\vec{A}' = U \vec{A} U^\dagger + i\hbar U \vec{\nabla} U^\dagger$:

$$\begin{aligned} \vec{A}' &= (2 - \sin^2 \theta) (\tilde{J}_x + \cot \theta J_z) \frac{\hat{e}_\varphi}{r} \\ &+ \frac{\cos \theta \sin^2 \theta}{\sqrt{1 + 2 \sin^2 \theta}} \left(\cot \theta \tilde{J}_x + \tilde{J}_y - J_z \right) \frac{\hat{e}_\varphi}{r} \\ &+ (\dots), \end{aligned} \quad (22)$$

where again (\dots) represents the terms which do not contribute to the singular part of the radial magnetic field. The only term describing a true $U(3)$ -monopole is

$$\vec{A}'_m = \frac{2 \cos \theta}{\sin \theta} \frac{\hat{e}_\varphi}{r} J_z \quad (23)$$

$$\vec{B}'_m = -2 \frac{\hat{e}_r}{r^2} J_z \quad (24)$$

with non-vanishing charge $Q = -2$.

B. Gauge transformations and magnetic charge

The previous situation is similar to the relationship between the $U(2)$ 't Hooft-Polyakov monopole and the $U(1)$ Dirac monopole [33]. A non-Abelian gauge transformation can be used to fully remove the string singularity of the Dirac monopole along the negative z axis by transferring it to the associated Higgs field. In the process, it is the total charge of the monopole and of the Higgs field that is conserved. For example, in Ref. [19], the Authors compute the ground state properties of a Bose-Einstein condensate subjected to a $U(2)$ -monopole (namely an Abelian one coupled to σ_x). They apply a gauge transformation to remove the string singularity (along the full z axis) to get a new non-Abelian gauge field but with a vanishing magnetic charge. This is the kind of situation we face here where we find a $U(3)$ -monopole with a vanishing charge in one gauge and with a non-vanishing charge in another gauge.

Let us illustrate a bit further the modification of the effective magnetic charge of the non-Abelian monopole

when a unitary change of the DS basis set is performed by discussing the usual tripod scheme studied in [12, 27]. It is obtained from our 2-tripod scheme by setting $\Omega_i^r = 0$. One can build a DS on the Zeeman states $|1\rangle$ and $|2\rangle$ coupled to the excited state $|6\rangle$ by the σ_+ and π Laguerre-Gauss beams. The DS basis set is then:

$$\begin{aligned} |D_1\rangle &= \frac{1}{\sqrt{2}} [e^{iS_{31}} |1\rangle - e^{iS_{32}} |2\rangle] \\ |D_2\rangle &= \frac{1}{\sqrt{2}} \cos \theta [e^{iS_{31}} |1\rangle + e^{iS_{32}} |2\rangle] - \sin \theta |3\rangle. \end{aligned} \quad (25)$$

Dropping the k -dependent part for simplicity, these DS generate the usual $U(2)$ -monopole gauge field, $\cos \theta \sigma_x \hat{e}_\phi / (r \sin \theta)$ which has an effective magnetic charge $Q = -1$.

If, on the contrary, one chooses to build a DS on the Zeeman states $|1\rangle$ and $|3\rangle$ coupled to the excited state by the σ_+ Laguerre-Gauss beam and by the σ_- Hermite-Gauss beam, one rather gets:

$$\begin{aligned} |D'_1\rangle &= \frac{1}{\sqrt{1 + \cos^2 \theta}} [\sqrt{2} \cos \theta e^{iS_{21}} |1\rangle - \sin \theta e^{iS_{23}} |3\rangle] \\ |D'_2\rangle &= \frac{1}{2\sqrt{1 + \cos^2 \theta}} [\sqrt{2} \sin^2 \theta e^{iS_{21}} |1\rangle + \\ &\quad 2 \sin \theta \cos \theta e^{iS_{23}} |3\rangle - \sqrt{2}(1 + \cos^2 \theta) |2\rangle]. \end{aligned} \quad (26)$$

Dropping again the terms which do not contribute to the singular part of the radial magnetic field, the new vector potential is:

$$\vec{A}' = \left[\mathbb{1} + \frac{\cos \theta \sin^2 \theta}{1 + \cos^2 \theta} \sigma_x + \frac{2 \cos^2 \theta}{1 + \cos^2 \theta} \sigma_z \right] \frac{\hbar \hat{e}_\phi}{r \sin \theta}, \quad (27)$$

where σ_x and σ_z are the standard Pauli matrices. These expressions bear some resemblance with the $U(3)$ situation, see Eq.(20). Here too, each term gives rise to a vanishing magnetic charge $Q = 0$. On the other hand, the transformation from the $|D'_a\rangle$ basis to the $|D_a\rangle$ basis ($a = 1, 2$) is given by the unitary matrix

$$U = \frac{e^{iS_{23}}}{\sqrt{1 + \cos^2 \theta}} \begin{pmatrix} \cos \theta & 1 \\ 1 & -\cos \theta \end{pmatrix}. \quad (28)$$

This shows that the gauge fields \vec{A}' and \vec{A} are simply related by the gauge transformation $\vec{A}' = U \vec{A} U^\dagger + i\hbar U \vec{\nabla} U^\dagger$. Therefore, the two gauge fields corresponds to the same physical situation, a $U(2)$ -monopole, but with $\vec{B}' = U \vec{B} U^\dagger$ despite the fact that the effective magnetic charges are different.

This is somehow what we have obtained for the $U(3)$ case above, i.e. the physics of a $U(3)$ -monopole but from an unusual gauge perspective. However it is important to note that, contrary to the $U(2)$ case, the $U(3)$ gauge potentials obtained with the laser configuration used for the alkaline-earth atoms and the gauge potentials obtained with the laser configuration used for the Rubidium case are *not* related by a gauge transformation. Indeed the

DS obtained in one scheme are linearly independent from the DS obtained with the other scheme since they span different subspaces of the full Hilbert space. Therefore they cannot be related by a (position-dependent) 3×3 unitary matrix, as required for a proper gauge transformation, only a larger one in the full Hilbert space can.

VII. CONCLUSION

In this paper we have proposed a workable experimental scheme, the so-called 2-tripod configuration, to generate non-Abelian $U(3)$ artificial static gauge fields for cold atomic gases. Our scheme only relies on one-photon resonant transitions and gives rise to three degenerate dark states. We have given the laser beams configurations for both alkali and alkaline-earth atoms and explained how to generate a $U(3)$ -monopole or a $U(3)$ spin-orbit coupling. Future work includes the study of the properties of the ground state and excitations of spinor condensates subjected to such $U(3)$ gauge potentials, in particular from a topological point of view (spin textures, etc).

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Appendix A: General expression for the scalar potential

The entries for the scalar potential Hermitian matrix Φ read:

$$\Phi_{11} = \frac{1 + \cot^2 \theta_r}{1 + \cot^2 \theta_l + \cot^2 \theta_r} \left[\frac{1}{4} \sin^2(2\phi_l) (\vec{\nabla} S_{12}^l)^2 + (\vec{\nabla} \phi_l)^2 \right]$$

$$\Phi_{33} = \frac{1 + \cot^2 \theta_l}{1 + \cot^2 \theta_l + \cot^2 \theta_r} \left[\frac{1}{4} \sin^2(2\phi_r) (\vec{\nabla} S_{12}^r)^2 + (\vec{\nabla} \phi_r)^2 \right]$$

$$\begin{aligned} \Phi_{22} = & \frac{\cot^2 \theta_l (1 + \cot^2 \theta_r)}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^2} \left(\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l \right)^2 \\ & + \frac{\cot^2 \theta_r (1 + \cot^2 \theta_l)}{(1 + \cot^2 \theta_r + \cot^2 \theta_l)^2} \left(\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r \right)^2 \\ & - 2 \frac{\cot^2 \theta_l \cot^2 \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^2} \left(\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r \right) \\ & \quad \times \left(\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l \right) \\ & + \left(\vec{\nabla} \frac{\cot \theta_l}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right)^2 \\ & + \left(\vec{\nabla} \frac{\cot \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right)^2 \\ & + \left(\vec{\nabla} \frac{1}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right)^2 \\ \Phi_{13} = & - \frac{\cot \theta_l \cot \theta_r}{1 + \cot^2 \theta_l + \cot^2 \theta_r} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) \times \\ & \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r + i \vec{\nabla} \phi_r \right) \end{aligned}$$

$$\begin{aligned} \Phi_{12} = & i \vec{\nabla} \left(\frac{\cot \theta_l}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right) \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) \\ & + \frac{\cot \theta_l (1 + \cot^2 \theta_l)}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{3/2}} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) \\ & \times (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l) \\ & - \frac{\cot \theta_l \cot^2 \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{3/2}} \left(\frac{1}{2} \sin(2\phi_l) \vec{\nabla} S_{12}^l - i \vec{\nabla} \phi_l \right) \\ & \times (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r) \end{aligned}$$

$$\begin{aligned} \Phi_{32} = & i \vec{\nabla} \left(\frac{\cot \theta_r}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{1/2}} \right) \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r - i \vec{\nabla} \phi_r \right) \\ & + \frac{\cot \theta_r (1 + \cot^2 \theta_l)}{(1 + \cot^2 \theta_l + \cot^2 \theta_r)^{3/2}} \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r - i \vec{\nabla} \phi_r \right) \\ & \times (\cos^2 \phi_r \vec{\nabla} S_{13}^r + \sin^2 \phi_r \vec{\nabla} S_{23}^r) \\ & - \frac{\cot \theta_r \cot^2 \theta_l}{(1 + \cot^2 \theta_r + \cot^2 \theta_l)^{3/2}} \left(\frac{1}{2} \sin(2\phi_r) \vec{\nabla} S_{12}^r - i \vec{\nabla} \phi_r \right) \\ & \times (\cos^2 \phi_l \vec{\nabla} S_{13}^l + \sin^2 \phi_l \vec{\nabla} S_{23}^l) \end{aligned}$$

Appendix B: The Gell-Mann matrices

The set of Gell-Mann matrices g_i ($1 \leq i \leq 8$) are one possible representation of the infinitesimal generators of the special unitary group $SU(3)$. An important representation features the 3×3 Hermitian λ -matrices $g_i = \lambda_i/2$

with:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

Appendix C: Scalar potential for the $U(3)$ case

The scalar potential for the laser scheme described by Eq. (5) (+ sign) or by Eq. (11) (− sign) reads:

$$\begin{aligned}
\Phi_{\pm} &= \frac{\hbar^2}{2m} \left[\left(\frac{1}{2r^2 \sin^2 \theta} + \frac{1}{2r^2} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right. \\
&\quad + \left(\frac{1}{r^2} + \frac{k^2}{2} \sin^2 2\theta \right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \mp \frac{\cos^2 \theta}{2r^2 \sin^2 \theta} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&\quad \left. - \frac{\sqrt{2}k}{4r} \sin 2\theta \sin \varphi \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix} \right]. \tag{C1}
\end{aligned}$$

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